Euler's "Mistake"? The Radical Product Rule in Historical Perspective

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1. INTRODUCTION. There is something pleasant in anecdotes about great mathematicians who made silly mistakes. A case in point: historians and mathematicians alike sometimes claim that Leonhard Euler, of all people, was confused about how to multiply imaginary numbers. His unlikely slips were published in his *Vollständige Anleitung zur Algebra* [*Complete Introduction to Algebra*] of 1770, widely esteemed as the best textbook on algebra of the eighteenth century. As the story goes, Euler thought that the product rule,

$$\sqrt{a} \times \sqrt{b} = \sqrt{(ab)} \tag{1}$$

is valid regardless of whether a and b are positive or negative. If the radicals in (1) mean the principal square-root operation, then Euler was wrong because, for example, $\sqrt{-4}\sqrt{-9} = -6 \neq \sqrt{-4 \times -9} = 6$. If instead we interpret the symbols to mean the unrestricted root operation, then Euler was still wrong, for $\sqrt{-4}\sqrt{-9} = \sqrt{4}\sqrt{9}\,(i^2) = \pm 6(-1) = \mp 6$, which is not equal to $\sqrt{(-4 \times -9)} = \pm 6$. Hence, for over two hundred years, writers have maintained that for negative numbers a and b the correct rule is

$$\sqrt{a} \times \sqrt{b} = -\sqrt{(ab)}. \tag{2}$$

One way or another, received wisdom has it, Euler got it wrong; he was just confused or mistaken (see, for example, [**17**, p. 121] or [**18**, p. 12]). Still, the matter is by no means as simple as it appears, and its history reveals intriguing subtleties in the axioms of algebra.

When Euler composed his *Algebra*, controversies still abounded regarding the rules on how to operate with negative and imaginary numbers. Such numbers were still often demeaned as "impossible." (For a history of some of the controversies surrounding "impossible" numbers before the 1750s, see [**22**].) In 1758, Francis Maseres had published his *Dissertation on the Use of the Negative Sign in Algebra* (as part of his bid for the Lucasian Chair of Mathematics at Trinity College), repudiating the use of isolated negative numbers and thus also of imaginaries. In 1765, François Daviet de Foncenex [**12**, p. 113] denounced as useless the representation of imaginary numbers as constituting a line perpendicular to a line of negatives and positives, a representation that decades later was used so successfully by Caspar Wessel, Jean-Robert Argand, and others. And, of course, Euler himself had been at the center of dispute on the question of the logarithms of negative numbers, in opposition to, among others, Jean d'Alembert [**5**] and Johann Bernoulli.

Euler defined mathematics as the science of quantity, where "quantity" signifies that which can be increased or decreased. Hence, imaginary numbers, being neither greater nor less than zero, were generally not considered quantities. (For a discussion of how, in connection with Euler's work, various kinds of numbers were considered fictitious instead of real quantities, see [**11**].) The question of how to multiply square roots of negative numbers was thus one muddle among many.

Any minor defects notwithstanding, Euler's *Algebra* was hailed as being "next to Euclid's *Elements*, the most perfect model of elementary writing, of which the scientific world is in possession" [**10**, p. xlviii]. Indeed, Euler's *Algebra* became one of the most widely read mathematics books in history, second only to the *Elements*. The *Algebra* was first published in Russian translation (two volumes, published in 1768 and 1769) before the standard German version appeared in 1770. In 1767 Euler was sixty years old and losing his eyesight. He dictated the book to a servant, a tailor's apprentice, so one might imagine that under such circumstances his account of the rules for the multiplication of roots contained simple mistakes or oversights, in what was otherwise a masterpiece. However, neither old age nor blindness slowed Euler's productivity or dulled his sharpness of mind, as is well known. Besides, Euler was increasingly acknowledged as the person who strikingly solved the vexing puzzle of taking logarithms of negative and complex numbers, as the latter eventually became known. So it seems stunning that he would have been confused about elementary multiplication. A passage in his *Algebra* seems even to state that $\sqrt{-1} \times \sqrt{-4} = 2$, ridiculous though it seems. Some historians, Florian Cajori for one [**4**, p. 127], have suggested that perhaps such errors stemmed merely from typographical printing miscues for which Euler cannot be held accountable. Others say that it involved a systematic confusion. Actually, neither is the case. Surprisingly, Euler committed no mistake on the matter. The solution to this puzzle is found buried in history under layers of ambiguous expressions, notations, and changing conventions on the definitions of basic operations. Stranger still, the historical analysis reveals defects and arbitrariness in the approach to this subject that became incorporated into elementary algebra as we know it.

2. EARLY OBJECTIONS TO THE PRODUCT RULE.

Around 1800, the symbol i was not yet widely used to stand for $\surd{-1}$ (though Euler had used it occasionally), and writers and typesetters used the signs $\surd$ and $\sqrt{}$ as equivalent, often meaning the *unrestricted* root operation. The *vinculum*, a separate bar adjacent to the radical sign, was used in the same way as a parenthesis, so that $\surd(ab) = \sqrt{ab} = \surd\,\overline{ab}$. Nowadays, both radical signs are commonly used to indicate that only the principal (or nonnegative) root should be extracted. In what follows, the meaning of each radical will be clear from context.

Before analyzing Euler's arguments, we begin by reviewing the arguments that were subsequently raised against the general validity of (1). Etienne Bézout, in the many editions of his *Cours de mathématiques*, gave reasons for rejecting Euler's claim. Bézout was an associate and *pensionaire* of the Paris Académie des Sciences, as well as a long-time teacher and examiner of would-be naval officers and other military personnel. In the 1781 edition of the *Cours*, Bézout discussed the multiplication of radicals as follows. He asserted rule (1): that to multiply radical quantities, one first multiplies the quantities as if there were no radical signs, and then extracts the root of the product. Yet, contrary to Euler, he then asserted the validity of (2):

$$\surd(-a) \times \surd(-b) = \surd(-a \times -b) = -\surd(ab),$$

and proceeded to justify it [**1**, p. 95, author's translation]:

> This last example deserves an explanation: according to the rule it might seem that $\surd(-a) \times \surd(-b)$ gives $\surd(-a \times -b)$, & consequently $\surd(+ab)$ or $\surd ab$; & every even radical being susceptible to two signs, $\pm$, one should have $\pm\surd ab$. But it must be observed that $\surd(-a) = \surd a \,.\, \surd(-1)$, & $\surd(-b) = \surd b \,.\, \surd{-1}$, therefore $\surd(-a) \times \surd(-b) = \surd a \,.\, \surd{-1} \,.\, \surd b \,.\, \surd{-1} =$

$\sqrt{a}\,.\,\sqrt{b}\,.\,\sqrt{(-1)}\,.\,\sqrt{(-1)} = \sqrt{ab}\,.\,\sqrt{(-1)^2}$; and $\sqrt{(-1)^2}$ is not indifferently ± 1, because the presence of the $-$ sign in $\sqrt{(-1)^2}$ makes it known by which operation one arrives at the square $(-1)^2$ of which to extract the root.

It is not entirely clear whether the expression $\sqrt{(-1)^2}$ was intended to convey $\left(\sqrt{(-1)}\right)^2$ or $\sqrt{\left((-1)^2\right)}$, though the final phrase suggests the latter. In any case, Bézout thus rejected the general validity of (1) on the grounds that $\sqrt{(-1)} \times \sqrt{(-1)} = -1$. He also claimed that $\left(\sqrt[n]{a}\right)^n = a$, saying "which is evident in general, if one realizes that the object is thus to return the quantity to its first state" [**1**, p. 96].

In an 1802 edition of Bézout's *Cours*, his arguments appeared in modified form. He again first stated the rule that, to multiply radical quantities, one first multiplies them as if there were no radical signs and then extracts the root of the product [**2**, p. 132]. However, he went on to write

$$\sqrt{a} \times \sqrt{a} = \sqrt{a^2} = a \tag{3}$$

and

$$\sqrt{-a} \times \sqrt{-a} = \sqrt{(-a)}^2 = -a. \tag{4}$$

Notice the ambiguous notation: Are the exponents inside the radicals or outside? Bézout added an explanatory footnote [**2**, p. 132, author's translation]:[1]

> We make, on this occasion, a remark that we consider very appropriate. Since $-a \times -a$ gives $+a^2$ of which the root is $\pm a$, $\sqrt{-a} \times \sqrt{-a}$ should therefore give $\pm a$; whereas here we give only $-a$. The reason is simple. When one demands what is the root of $+a^2$, one has reason to assign equally $+a$ and $-a$, because nothing in that question specifies whether one considers $+a^2$ as coming from $+a \times +a$, or from $-a \times -a$. But when one demands what is the value of $\sqrt{-a} \times \sqrt{-a}$, since this quantity, according to the rules, reduces to $\sqrt{+a^2}$, one should only obtain $-a$, because here the question itself fixes from which operation $+a^2$ comes. By paying attention to this, one will see that $\sqrt{-a} \times \sqrt{-b}$ shall give $-\sqrt{ab}$; and not $\pm\sqrt{ab}$; because, $\sqrt{-a}$ being the same thing as $\sqrt{a} \times \sqrt{-1}$, and $\sqrt{-b}$ being the same as $\sqrt{b} \times \sqrt{-1}$, [therefore] $\sqrt{-a} \times \sqrt{-b}$ will be $\sqrt{a} \times \sqrt{b} \times \sqrt{-1} \times \sqrt{-1}$, or $\sqrt{ab} \times \sqrt{(-1)^2}$, which is $-\sqrt{ab}$, because $\sqrt{(-1)^2} = -1$.

Hence Bézout distinguished between cases where the extraction of square roots yields two solutions or one. He admitted two solutions, on the one hand, by writing "$x^2 = 25$, $x = \pm 5$," and "$x^2 = -4$, $x = \pm\sqrt{-4}$," so that positive and negative numbers both have two square roots. On the other hand, he claimed that if we know that a number comes from the multiplication of two given numbers, as in $5 \times 5 = 25$, then there is only one root, so $\sqrt{5} \times \sqrt{5} = \sqrt{25} = 5$. The conclusion seems reasonable, although we may not be satisfied with Bézout's argument. In particular, he settled the matter by introducing a rule precisely to ensure that $\sqrt{(a^2)} = a$ when we know the value of a. Without this independent rule one might otherwise expect that, in accordance with the definition of the unrestricted radical operation, $\sqrt{(a^2)} = \pm a$.

[1]The text is riddled with typographical errors and idiosyncrasies: two different signs designate radicals though both refer to exactly the same operation, in one place the multiplication sign is mistakenly replaced with an addition sign, and so forth. My translation corrects such defects, just as later editions corrected the passage, including two significant clarifications: the word "therefore" inserted in brackets and rewriting $\sqrt{(-1)}^2$ as $\sqrt{(-1)^2}$ so that the exponent is inside the radical [**3**, p. 98].

In any case, other mathematicians agreed that Euler's account was defective. Sylvestre François Lacroix, for example, agreed with Bézout's arguments. Lacroix was a renowned professor of mathematics at the École Centrale des Quatre-Nations, as well as the successor to Lagrange's chair at the École Polytechnique. In Article 164 of his influential *Élémens d'algèbre*, Lacroix asserted the following rule: $\sqrt[x]{a} \times \sqrt[x]{b} = \sqrt[x]{ab}$ [**16**, p. 233]. Yet he commented on "certain singular cases" of such rules that could "lead to error in regard to imaginary quantities, if one does not accompany them with remarks that pertain to the properties of two terms." He continued [**16**, pp. 239–240, author's translation]:

> For example, the rule in Article 164 gives immediately
>
> $$\sqrt{-a} \times \sqrt{-a} = \sqrt{-a \times -a} = \sqrt{a^2};$$
>
> and if one contents oneself with taking $+a$ for $\sqrt{a^2}$, the result will be visibly faulty, because the product $\sqrt{-a} \times \sqrt{-a}$, being the square of $\sqrt{-a}$, should be obtained by suppressing the radical, and consequently is equal to $-a$.
>
> Bézout has explained this difficulty very well, by observing that when one does not know how the square a^2 has been formed, and one seeks its root, one should well assign equally $+a$ and $-a$; but when one knows in advance which of these two quantities has been multiplied by itself to form a^2, it is then no longer allowed, as one returns on one's steps, to take another. This is evidently the case in the expression $\sqrt{-a} \times \sqrt{-a}$, where one knows therefore that the quantity a^2, contained under the radical $\sqrt{a^2}$, comes from $-a$ multiplied by $-a$; therefore the ambiguity ceases, and when one returns to the root, one must put $-a$.
>
> The same embarrassment would take place also for the product $\sqrt{a} \times \sqrt{a}$, if one were not driven, since there is no $-$ sign in the expression, to take immediately the positive value of $\sqrt{a^2}$. One must see that, in this case, a^2 comes from $+a$ multiplied by $+a$, so its root must be $+a$.

Thus Lacroix agreed with (3) and (4) of Bézout. Their texts were widely published, revised, and reprinted for many years, including translations into German, English, Spanish, Italian, and Russian.

Their arguments convinced most mathematicians, yet there remained some hesitations about the matter. For example, Jeremiah Day, President of Yale College, noted in his very popular algebra textbook that "I have been unwilling to admit into the text rules of calculation which are commonly applied to imaginary quantities; as mathematicians have not yet settled the logic of the principles upon which these rules must be founded" [**6**, pp. 324–325]. In particular, he noted that Euler and others had asserted that $\sqrt{-a} \times \sqrt{-a} = \pm a$, whereas, like Bézout and Lacroix, Day argued that the result should be not $+a$ or $-a$, but exclusively $-a$. By 1845 at least ten French editions of Bézout's *Cours* had been published, and more editions followed. Moreover, by 1868 Lacroix's algebra textbook had seen twenty-two French editions. Subsequently, other textbooks likewise conveyed consensus with the arguments advocated by Bézout and Lacroix. The conclusion spread: Euler had made an embarrassing mistake in his *Algebra*.

3. EULER'S TREATMENT OF THE PRODUCT RULE. What did Euler actually claim about the multiplication of radicals? The answer is not straightforward, because *nowhere* in the *Algebra* did he even write the equations $\sqrt{a} \times \sqrt{b} = \sqrt{(ab)}$, $\sqrt{-a} \times \sqrt{-a} = \sqrt{(a^2)}$, etc. He discussed the topic first without using the "=" sign. Rather than formulating such rules as equations, he expressed them in words and by means of examples. Euler's *Algebra* was not a rigorous deductive treatise in which he expressed each proposition in its most general and exacting form. Instead, it was a didactic work intended for students wherein he introduced propositions gradually. Accordingly, we must treat his expressions with care.[2]

Consider Euler's initial statements. In Article 131 he claimed: "the square-root of 2 multiplied by itself must give 2, so we know that the multiplication of $\sqrt{2}$ by $\sqrt{2}$ necessarily produces 2... and in general $\sqrt{a}$ multiplied by $\sqrt{a}$ gives a" [**8**, p. 79]. Thus he seems to have stated that $\sqrt{a} \times \sqrt{a} = a$, the very rule that later mathematicians cited to establish that (1) is not generally true! But notice: Euler did *not* use the equality sign, so we might wonder whether "to produce" or "to give" a result a is the same as to produce *only* a. Such wording was not unique. In Article 328 he claimed that "$\sqrt{a}$ multiplied by $\sqrt{a}$ gives a" [**8**, p. 205] and likewise for negatives in Article 329 that "$\sqrt{-a}$ multiplied by $\sqrt{-a}$ gives $-a$" [**8**, p. 206], *instead* of writing equations, even though he had introduced the "=" sign long before, namely, in Article 206. As remarked above, some early writers (such as Day) understood Euler to have claimed that $\sqrt{a} \times \sqrt{a} = \pm a$. For instance, in Article 122 he stated that "from every square are given two square-roots, of which one is positive, the other negative" [**8**, p. 72]. Is that what he meant throughout?

In Article 132 Euler first stated (1): "when it is required to multiply $\sqrt{a}$ by $\sqrt{b}$, the product is $\sqrt{ab}$" [**8**, p. 79] (clearly, the notation meant $\sqrt{(ab)}$ rather than $\sqrt{a}b$). The same rule reappeared in Article 328, again without the equality sign: "$\sqrt{a}$ multiplied by $\sqrt{b}$ gives $\sqrt{ab}$ ..." [**8**, p. 205]. What did this rule entail when applied to negatives?

We turn to his initial statements on how to operate with "imaginary quantities," in chapter 13 (of part 1) of the *Algebra*. Consider Article 146 [**8**, p. 86]:

> 146. Now, first of all, what we agree about impossible numbers, e.g. of $\sqrt{-3}$, consists of this: that the square of it, or the product that results when $\sqrt{-3}$ is multiplied by $\sqrt{-3}$, gives -3, so also $\sqrt{-1}$ multiplied by $\sqrt{-1}$ is -1. And in general that multiplying $\sqrt{-a}$ by $\sqrt{-a}$, or taking the square of $\sqrt{-a}$, gives $-a$.

Thus Euler began with what seems to be the specific rule that Bézout, Lacroix, and others appealed to in proving that (1) is not valid for negatives, that $\sqrt{-a} \times \sqrt{-a} = -a$. Take note, however, that Euler was still *not* using the equality sign. Significantly, when Euler introduced the equality sign in Article 206, he defined it to stand in place of the expressions "is as much as" and "is equal to," neither of which he used when verbalizing rule (1) in Articles 131, 132, 328, 329, and 146.[3] He continued in the next article:

> 147. Since $-a$ is as much as $+a$ multiplied by -1, and since the square-root of a product is found when we multiply together the square-roots of its factors, so is the radical of a multiplied by -1, or $\sqrt{-a}$, so much as $\sqrt{a}$ multiplied by $\sqrt{-1}$. Now, since $\sqrt{a}$ is a possible number, consequently, the impossibility may always be reduced to $\sqrt{-1}$. On these grounds $\sqrt{-4}$ is just as much as $\sqrt{4}$ multiplied by $\sqrt{-1}$: or since $\sqrt{4}$ is 2, therefore $\sqrt{-4}$ is as much as $2\sqrt{-1}$. And $\sqrt{-9}$ is as much as $\sqrt{9}.\ \sqrt{-1}$, that is, $3\sqrt{-1}$. And $\sqrt{-16}$ is as much as $4\sqrt{-1}$.

Here Euler justified the practice of writing $\sqrt{-a}$ as $\sqrt{a}\sqrt{-1}$ precisely on the basis of (1), that "the square-root of a product is found when we multiply together the square-

[2]The English translation of Euler's *Algebra* was made from the French translation. I give my own, more literal, translation of Euler's original German. All subsequent references are thus to the original German edition of 1770 unless otherwise noted.

[3]In Euler's words: "... brauchen wir ein neues Zeichen, welches anstatt der bisher so häufig vorgekommenen Redens-Art, ist so viel als, gesetzt werden kann. Dieses Zeichen ist nun = und wird aus gesprochen ist gleich" [**8**, p. 123] (emphasis in the original).

roots of its factors," thus claiming its validity for negative as well as positive numbers. Here he *did* use the equality expression "is as much as." Of course, in this instance he referred only to "mixed" factors, as in $\sqrt{-4} = (\sqrt{4}) \times (\sqrt{-1}) = 2\sqrt{-1}$, where the multiplied radicands 4 and -1 have opposite signs. It was not yet clear whether $\sqrt{-4}$ was to have two values or one, because the radical in $2\sqrt{-1}$ could be construed to indicate two solutions. Immediately, he elucidated the situation [**8**, pp. 87–88]:

> 148. Moreover, as $\sqrt{a}$ multiplied by $\sqrt{b}$ gives $\sqrt{ab}$, so too will $\sqrt{-2}$ multiplied by $\sqrt{-3}$ give $\sqrt{6}$. Likewise, $\sqrt{-1}$ multiplied by $\sqrt{-4}$ gives $\sqrt{4}$, that is, 2. Thus we see that two impossible numbers, multiplied together, yield a possible or real one.

Again, Euler referred to (1) in order to establish the rule for the product of the radicals of negatives.

Yet here there are some apparent embarrassing mistakes. For example, later editors of Euler's *Algebra* have noted critically: "We should set $\sqrt{-2} \cdot \sqrt{-3} = \sqrt{2} \cdot \sqrt{3} \cdot (\sqrt{-1})^2 = -\sqrt{6}$" [**20**, p. 87]. Likewise, Tristan Needham in his *Visual Complex Analysis* (1997) comments that "in 1770 the situation was still sufficiently confused that it was possible for so great a mathematician as Euler to mistakenly argue that $\sqrt{-2}\sqrt{-3} = \sqrt{6}$" [**19**, p. 1]. Historian Ivor Grattan-Guinness also notes: "Euler gave a reliable presentation; but he gaffed in his algebraic handling of complex numbers, by misapplying the product rule for square roots, $\sqrt{(ab)} = \sqrt{a}\sqrt{b}$, to write $\sqrt{-2}\sqrt{-3} = \sqrt{6}$ instead of $-\sqrt{6}$" [**14**, p. 334].

Moreover, Euler seems to have asserted that $\sqrt{-1} \times \sqrt{-4} = \sqrt{4} = 2$. In this case, too, recent writers have claimed that he erred, that the correct result is -2 (e.g., Kline [**17**, p. 121], Nahin [**18**, p. 12]). But did Euler deem $+2$ to be the *only* solution? Earlier, in Article 122, he had emphasized that every square has two square-roots, one positive and the other negative. Of course, 4 is a perfect square. More importantly, throughout Articles 115 to 120, in which he introduced the concepts of squares and roots, Euler repeatedly spoke of squares as having but one root (e.g., that "2 is the square-root of 4" [**8**, p. 68]). Evidently in those articles he made no "mistake" on the matter, but was simply introducing the subject by considering first only the positive roots. This raises the question: Did he mean that the expression $\sqrt{-1} \times \sqrt{-4}$ "gives" *only* a positive value? The answer was provided shortly thereafter [**8**, p. 88]:

> 150. Since, again, the aforesaid remark always takes place, that the square-root of a given number always has a double value, thus negative as well as positive can be taken, in that e.g., $\sqrt{4}$ is $+2$ as well as -2, and in general, for the square-root of a we can write $+\sqrt{a}$ as well as $-\sqrt{a}$, so this holds also for the impossible numbers; and the square-root of $-a$ is $+\sqrt{-a}$ as well as $-\sqrt{-a}$ . . . .

The placement of this passage in the context of his argument strongly suggests that Euler ascribed two values to square-roots obtained from (1). Whereas other mathematicians argued that not all square-root radicals entail two values, precisely in discussing the applicability of (1), Euler claimed that all square-root radicals do have two roots, right *after* discussing the rule in question. He stated that this rule is "always" valid. Also, his expression "the square-root of a given number always has a double value" suggests that his frequent use of the singular phrase "the square-root" did not mean that only single solutions are obtained.

He made the same distinction between "the square root" and its "values" at other points in the *Algebra*, for example, in Article 69 of part 1 and in Articles 79 and 139 of part 2.[4] Euler only explicitly rejected nonpositive roots in the context of physical applications. For instance, regarding square roots he wrote: "[A]ll these questions admit of a double solution, but in some cases, where perhaps the question pertains to a certain number of men, the negative value is discarded" [**8**, pt. 2, p. 63], [**10**, pt. 1, Article 631, p. 219]. Since his book was designed as a teaching manual, many problems in it admitted only positive solutions as physically meaningful. Yet the double solutions were valid algebraically. Finally, in other writings Euler also claimed that there are two solutions to every square-root radical. A case in point is his "Researches on the Imaginary Roots of Equations" of 1749, in which he emphasized that the quantity of imaginary roots is *always* even and *never* odd and that "by its nature the radical sign encompasses essentially the $+$ sign as well as the $-$ sign," that is, two solutions [**7**, pp. 80–81, 113].

We conclude that Euler intended to assert that

$$\sqrt{a} \times \sqrt{b} = \sqrt{-a} \times \sqrt{-b} = \sqrt{(ab)}$$

and, in particular, that

$$\sqrt{a} \times \sqrt{a} = \sqrt{-a} \times \sqrt{-a} = \sqrt{(a^2)} = \pm a.$$

4. ANALYSIS OF EULER'S POSITION. Only now can we analyze whether there were errors or inconsistencies in Euler's position. Consider again the example $\sqrt{-4} \times \sqrt{-9} = \sqrt{(-4 \times -9)}$, where now we assert the equality in accordance with Euler's claims. Contrary to the arguments of Lacroix and Bézout, we may disregard any notions that the right side of this equation "comes from" the left side, for we might just as well say the opposite. We can look at the equation without imposing upon it any preconceived temporal sequence, of saying which came "first." Simply, two expressions given simultaneously are separated by the equality sign, and we wish to ascertain whether they are really equivalent. We simplify each expression directly. Consider the right side, $\sqrt{(-4 \times -9)}$. By multiplying first, we obtain $\sqrt{36}$, after which the unrestricted radical operation yields ± 6. In Euler's words, "the sign $\pm$ is read *plus* or *minus* and indicates that such terms can be sometimes positive and sometimes negative" [**8**, pt. 2, p. 59], [**10**, pt. 1, Article 626, p. 217]. Now turn to the left side of the equation, $\sqrt{-4} \times \sqrt{-9}$. We extract the square roots first to obtain $\pm 2i \times \pm 3i$, where the term $\pm 2i$ indicates that there are two imaginary solutions for $\sqrt{-4}$ and $\pm 3i$ indicates two imaginary solutions for $\sqrt{-9}$. A pair of double signs was used systematically by Euler to represent four values [**8**, pt. 2, p. 165], [**10**, pt. 1, Article 753, p. 272]. Accordingly, we may have:

$$(+2i) \times (+3i) = -6, \qquad (+2i) \times (-3i) = +6,$$
$$(-2i) \times (+3i) = +6, \qquad (-2i) \times (-3i) = -6.$$

These results are summarized by stating that $(\pm 2i) \times (\pm 3i) = \pm 6$. Thus, by interpreting things as we have, we conclude (as we believe Euler intended) that

$$\sqrt{-4} \times \sqrt{-9} = \sqrt{(-4 \times -9)}.$$

[4] [**8**, pt. 1, p. 63 (Article 69); pt. 2, pp. 71–72, 117–118 (Articles 79 and 139, respectively)]. In the original German edition of 1770, the articles of part 1 are numbered from 1 to 562. However, some later editions, especially the French and English ones, included many articles from part 2 in part 1 so that the numbering is different. Accordingly, the original Articles 79 and 139 of section 1 of part 2 appear as Articles 641 and 701, respectively, of section 4 of part 1 of the English edition.

This same procedure yields the same result if the numbers involved are positive. In this way, surprisingly, the equation $\sqrt{a} \times \sqrt{b} = \sqrt{(ab)}$ is valid for both positive and negative numbers. In short Euler's approach is correct.

An apparent objection: Is this procedure really independent of the rule that we seek to demonstrate? In particular, by writing $\pm 2i$ for $\sqrt{-4}$, it might appear that we need to assume that $\sqrt{-4} = \sqrt{(4 \times -1)} = \sqrt{4} \times \sqrt{-1}$, that is, rule (1). However, in the case of a positive number we have, for example, $\sqrt{4} = \pm 2$, so the rule $\sqrt{a} \times \sqrt{b} = \sqrt{(ab)}$ can first be obtained by considering the properties of positives. It may then be applied, by extension, to negatives just as other rules governing positive numbers were deemed correct as well for negatives. Indeed, that is precisely how Euler introduced the product rule in his *Algebra*. But even if we admit the validity of (1) for negative numbers as merely an independent axiom, its application still leads to entirely consistent results, whence there is then no error in asserting its truth for both positives and negatives. In any event, we justified the procedure in question by appealing directly to the independent rule that every square-root radical sign corresponds to two values, so that, for example, $\sqrt{-4} = \pm 2i$. (Here, the symbol i designates a root of the polynomial $x^2 + 1 = 0$, whereby $\sqrt{(-1)} = \pm i$.)

Another potential objection concerns the "$\pm$" symbol. Leaving history aside for a moment, suppose that someone now were to argue that the product of $\pm 2i \times \pm 3i$ could *only* be expressed by ∓ 6, not by ± 6. If so, then not (1) but (2) would seem to be correct:

$$\sqrt{-4} \times \sqrt{-9} = -\sqrt{(-4 \times -9)}$$

would translate to

$$\pm 2i \times \pm 3i = -\sqrt{36},$$

whence

$$\mp 6 = -(\pm 6)$$

or

$$\mp 6 = \mp 6.$$

(Unlike the disjunction operator of ordinary logic, the "or" of numerical signs is not commutative, so that "plus or minus" is not equal to "minus or plus": $\pm a \neq \mp a$.) There are good reasons to consider this argument, since even Euler used the symbols in question in the following way: $-1 \times \pm a = \mp a$ (see, for example, Article 140 [**8**, pt. 2, p. 118] or Article 702 in [**10**, pt. 1, p. 247]).[5] Nonetheless, some observations are in order. First, Euler everywhere expressed the signs of the values of $\sqrt{(a)}$ as $\pm$, regardless of whether a was positive or negative. Only afterwards in the text did the double sign sometimes change to $\mp$, specifically when transposing terms from one side of an equation to another. Second, the question of how results such as -6, $+6$, $+6$, -6, may be abbreviated is decided by convention, and there is nothing in Euler's *Algebra* showing that they should be abbreviated by the $\mp$ symbol. Moreover, this sort

[5]The symbol $\mp$ appears only a very few times in the 1770 edition, so its original use by Euler must be analyzed with this in mind. Note, for example, that the symbol was inserted into Article 629 of the English edition, whereas it is absent in the corresponding original, namely, Article 67 [**8**, pt. 2, p. 61]. Likewise, it appears in the original Article 76 [**10**, pt. 2, p. 69], but it was replaced with $\pm$ in the English edition's Article 638 [**10**, pt. 2, p. 222].

of requirement about the order of terms is something that may be introduced from various perspectives. For instance, to suspend history again, someone might argue that since $\sqrt{4} = \pm 2$, we should also have: $\sqrt{-4} = \mp 2i$. Accordingly, if we stipulate that $\sqrt{(-4 \times -4)} = \mp 4$, we might also well allow $\sqrt{(-4 \times -9)} = \mp 6$, in which case (1) is again valid for negatives as well as positives. But such considerations are arbitrary. What matters for the historical question is that there be a straightforward account in which Euler's statements can be understood as coherent. Here the basic procedure suffices, because by admitting both values of each radical and employing the fourfold multiplication of signs, (1) gives the same results when applied to any pair of negative numbers as it does for the corresponding pair of positives. It also works for any combination of negatives and positives. By the way, the same procedure explains Euler's rules on the *division* of imaginaries, which have *also* been deemed erroneous by modern writers such as Cajori [**4**, p. 127] and Grattan-Guinness [**14**, p. 335].

We return now to the reasons why mathematicians thought Euler had committed a mistake. The general validity of (1) was first rejected on the grounds that

$$\sqrt{a} \times \sqrt{a} = \sqrt{(a^2)} = a. \tag{5}$$

However, Euler's approach has a definite advantage over that of Bézout, Lacroix, and others. Only in Euler's approach does the following axiom apply universally: equal operations performed on both sides of an equation always preserve the equality. As the simplest example, consider the identity $(+a)^2 = (-a)^2$, and extract square roots. If we respect the rule that every square-root radical has two values, then $\sqrt{((+a)^2)} = \sqrt{((-a)^2)}$, so $\pm a = \pm a$. Otherwise, according to the rule of Bézout and Lacroix, we would obtain $+a = -a$, a clear contradiction (unless $a = 0$). Thus, Euler's approach was superseded by an approach that tacitly violated one of the most elementary rules of arithmetic. In this light, it was not Euler at all who was just plain wrong. At stake was precisely the question of whether all the rules of arithmetic should hold in what later became known as symbolic algebra.

In the long run, Bézout's rule (5) was not the justification that mathematicians invoked widely to restrict the validity of (1). Consider another early argument against Euler, a commentary written by J. G. Garnier, Professor of the École Polytechnique, and appended as a critical note to chapter 13 of Euler's *Algebra* in an 1807 French edition. Garnier explained that: "To multiply $\sqrt{-1}$ by $\sqrt{-1}$ is to take the square of $\sqrt{-1}$; it is therefore to return to the quantity that is under the radical. Therefore, one has $\sqrt{-1} \times \sqrt{-1} = -1$" [**13**, p. 498]. He then cautioned [**13**, pp. 498–499]:

> One cannot say that $\sqrt{-1} \times \sqrt{-1} = \sqrt{+1} = \pm 1$, as one might conclude from the rule for multiplying two radicals having the same index. Because, supposing that $x = \sqrt{-1}$, then, if it were possible, $x^2 = \pm 1$; one would have, by taking the superior sign, and extracting the square root of that part and of the other, $x = \sqrt{-1} = \pm 1$, that which is absurd, because an imaginary number would be equal to a real number.

Thus Garnier rejected the general correctness of (1). His argument may be clarified. For him, $x^2 = \pm 1$ stood for $x^2 = +1$ or $x^2 = -1$. Extracting the square roots in the former (positive) equation: $\sqrt{(x^2)} = \sqrt{+1} = \pm 1$, and "supposing that $x = \sqrt{-1}$," yields the "absurd" result that he wrote as $x = \sqrt{-1} = \pm 1$. The apparent contradiction, however, arose only because Garnier chose to correlate $x = \sqrt{-1}$ to the positive value of $x^2 = \pm 1$, whereas it corresponds only to the negative value. In that case, $\sqrt{(x^2)} = \sqrt{-1} = x$, whereas, $x^2 = +1$ corresponds to $\sqrt{1}$, as $\sqrt{(x^2)} = \sqrt{+1} = \pm 1 = x$, and there is no absurdity. In any case, like Bézout, Garnier seems to have affirmed

(3). More precisely, he asserted that

$$\sqrt{a} \times \sqrt{a} = (\sqrt{a})^2 = a \tag{6}$$

instead of (5), at the general expense of (1). Rule (6) requires that the product of equal radicands be equal to one radicand squared and that, in *this* case, the root and square operations be *exactly inverse* to one another, resulting in the original radicand. Whereas, following Bézout, rule (5) was construed as a case of (1), rule (6) is clearly independent of (1).

Rather than asserting the ambiguous equation (3), mathematicians established narrower rules for the multiplication of radicals. Some writers, such as Lacroix, asserted rule (5). Later, many others, such as George Peacock, adopted rule (6) in its place [**21**, p. 72]. Euler's approach constitutes a third alternative. Equations (3), (4), (5), and (6) all circumvent the rule that *every* nonzero square-root has two values. To take the simplest example, (3) requires that $\sqrt{1} \times \sqrt{1} = 1$, but if we actually extract the square roots and apply the fourfold multiplication of "plus or minus," we obtain

$$\sqrt{1} \times \sqrt{1} = \pm 1 \times \pm 1 = \begin{bmatrix} +1 \times +1 = +1 \\ +1 \times -1 = -1 \\ -1 \times +1 = -1 \\ -1 \times -1 = +1 \end{bmatrix} = \pm 1,$$

contrary to Garnier's claims. Hence, depending on how we define the multiplication of radicals, results differ. Accordingly, although many mathematicians sided with Bézout and Lacroix in regarding the equation $\sqrt{-1} \times \sqrt{-1} = -1$ as necessarily and exactly true, certain mathematicians later in the nineteenth century deemed it merely a useful "convention" or "supposition." These include the Cambridge professors Isaac Todhunter [**24**, pp. 213–214] and Charles Smith [**23**, p. 221]. Of course, they had the benefit of enlightenment on the role of arbitrary conventions in the foundations of algebra thanks to the development of symbolic algebra at the hands of Peacock and Augustus De Morgan.

As with the multiplication of equal radicals, the squaring of a radical involves a supposition concerning its values. Depending on how we define the operation of squaring the "±" sign, we get different results. For instance, since we commonly allow that $\sqrt{(a^2)}$ has two values, we have $\sqrt{(a^2)} \neq (\sqrt{a})^2$, because we suppose also that

$$\sqrt{a} \times \sqrt{a} = (\sqrt{a})^2 = (\pm r)^2 = \begin{bmatrix} (+r)^2 = (+r) \times (+r) = +a \\ or \\ (-r)^2 = (-r) \times (-r) = +a \end{bmatrix} = +a, \tag{7}$$

where $r^2 = a$. Here we have multiplied only each square root by itself. This is just what Garnier argued in his note to Euler's text: one should have

$$(\pm r)^{2m} = \left((\pm r)^2\right)^m = (+r^2)^m = +r^{2m},$$

that is, only the positive solution [**13**, p. 491]. Thus, $\sqrt{(a^2)} \neq (\sqrt{a})^2$ became the norm. For example, Charles Smith, master of Sidney Sussex College of Cambridge University, explained: "It should be remarked that it is not strictly true that $\sqrt[n]{(a^m)} = (\sqrt[n]{a})^m$ ... unless by the nth root of a quantity is meant only the *arithmetical* root. For example, $\sqrt[2]{(a^4)}$ has two values, namely $\pm a^2$, whereas $(\sqrt[2]{a})^4$ has only the value $+a^2$" [**23**, p. 206]. By contrast, we may otherwise get $\sqrt{(a^2)} = (\sqrt{a})^2$ by supposing instead that

$$(\sqrt{a})^2 = (\pm r)^2 = (\pm r) \times (\pm r) = \left[\begin{array}{c} (+r) \times (+r) = +a \\ (+r) \times (-r) = -a \\ (-r) \times (+r) = -a \\ (-r) \times (-r) = +a \end{array} \right] = \pm a. \qquad (8)$$

Here we have multiplied each square root by itself and the other. The result is consonant with Euler's product rule, but it seems quite unnatural because its fourfold multiplication clashes with the common notion of squaring as involving only the multiplication of identical terms. Still, it would be symbolically plausible if only we were willing to define the operation of squaring strictly as multiplication (including the fourfold case), rather than define it as a distinct operation. In both (7) and (8), squaring is defined in terms of multiplication, yet the results are distinct. (Further still, another alternative would be to define squaring independently of the fourfold multiplication, such that $\pm a = \sqrt{a} \times \sqrt{a} \neq (\sqrt{a})^2 = a$, where Euler's product rule would yet hold.)

The reactions against Euler's account stemmed partly from notions that now are considered to be nonmathematical. In particular, algebra was often treated as involving relationships with specific order in time. Some philosophers, such as Kant, intimately associated notions of numerical order with temporal order. Attempting to provide a grounding for signed numbers, W. R. Hamilton came to construe algebra as the "Science of Pure Time" [**15**]. Alongside such overt formulations, the words of algebraists were permeated with temporal notions (e.g., terms such as "root" and "product" could seem to presuppose temporal ordering; one side of an equation was sometimes said to precede the other; knowledge of what came from where was construed to decide the acceptability of certain solutions). Mathematicians gradually abandoned these metaphysical and epistemological perspectives. Yet there remained rules, such as (2) and (5), that had been introduced partly on the basis of such notions. If only Euler's product rule had been properly understood, it might have appeared admissible, because it preserved arithmetic relations. But that is hypothetical. The fact is that mathematicians increasingly turned away from the idea that algebra should conform to arithmetical forms, just as physical and geometrical analogies had earlier been rejected as justifications for rules.

In sum, most algebraists did not grasp Euler's approach because it clashed with other rules that they posited. More recently, writers have overlooked Euler's explanation, missing the point that for him the word "gives" did not mean "="—and that the consequent double-valued square-roots imply that the rules (4), (5), (6), and (7) are not indispensable. Such rules are discarded if we adopt instead Euler's alternative.

In the end, it comes down to a choice of axioms. If we assume that all square roots have two values *and* require the fourfold multiplication of double signs, then Euler's results are justifiable. Otherwise, the product rule can be restricted by positing independent rules like (5), (6), or (7). This restriction trades economy and generality of axioms for the convenience of simpler results. It has the advantage of reducing the proliferation of the ambiguous "±" sign. Above all, it leads to the immensely useful principle of exponents: $a^{1/2} \times a^{1/2} = a^{1/2+1/2} = a^1 = a$. However, Euler's approach has certain advantages over the approach that became widespread. It ensures the commutativity of the unrestricted radical and squaring operations. It admits into symbolic algebra certain unrestricted arithmetical properties, such as (1) and $(\sqrt{a})/(\sqrt{b}) = \sqrt{(a/b)}$, that would otherwise be absent.

If one ignores the plasticity of the elements of symbolic algebra, Euler's statements on the rules of signs and radicals seem mistaken. After all, before a system of laws was universally adopted, mathematicians had some freedom in choosing whatever axioms they saw fit, with the result that symbolic algebra could develop in different directions. Euler's account of the multiplication of radicals was rejected not only because it was not clearly understood, but also because it clashed with properties that mathematicians preferred in the algebra of signed numbers. Over time, $\sqrt{-1}$ served increasingly to signify the single numerical value i rather than $\pm i$, while the "$\sqrt{\ }$" sign was used more and more to designate only nonnegative roots. Such conventions simplified elementary algebra, eliminating ambiguities of multiple solutions that otherwise complicate even basic calculations [**21**, pp. 74–76]. Nevertheless, Euler's alternative approach reminds us that even the axioms of elementary algebra admit subtle variations that can lead to alternative algebraic structures.

ACKNOWLEDGMENTS. I thank Chandler Davis, Craig G. Fraser, Akihiro Kanamori, and Christopher Smeenk for helpful comments. I also thank the anonymous referees.

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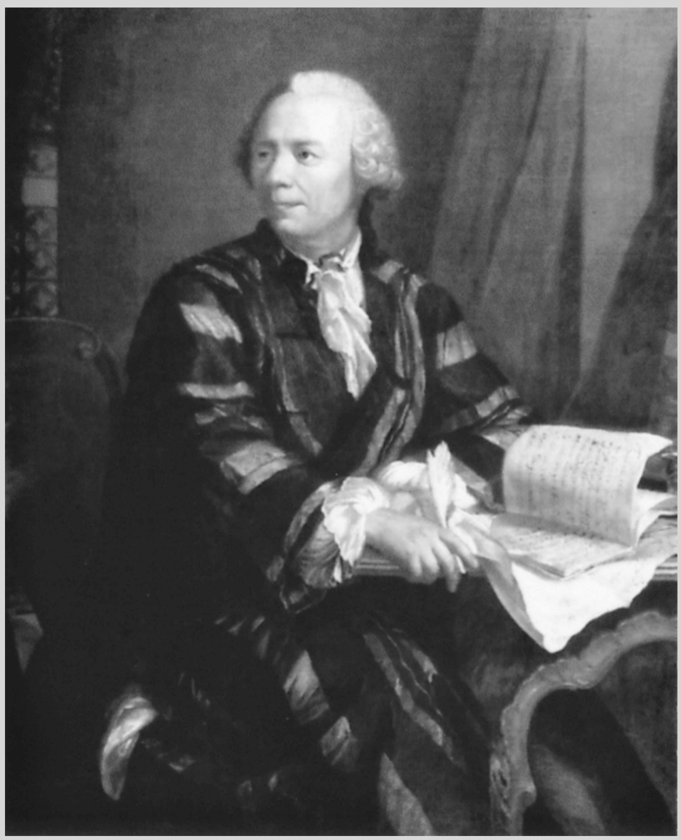
Happy 300th Birthday, Euler!



Leonhard Euler, April 15, 1707 – Sept. 18, 1783.